

# Supersymmetric isolated horizons

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## Abstract

We construct a covariant phase space for rotating weakly isolated horizons in Einstein-Maxwell-Chern-Simons theory in all (odd)  $D \geq 5$  dimensions. In particular, we show that horizons on the corresponding phase space satisfy the zeroth and first laws of black-hole mechanics. We show that the existence of a Killing spinor on an isolated horizon in four dimensions (when the Chern-Simons term is dropped) and in five dimensions requires that the induced (normal) connection on the horizon has to vanish, and this in turn implies that the surface gravity and rotation one-form are zero. This means that the gravitational component of the horizon angular momentum is zero, while the electromagnetic component (which is attributed to the bulk radiation field) is unconstrained. It follows that an isolated horizon is supersymmetric only if it is extremal and nonrotating. A remarkable property of these horizons is that the Killing spinor only has to exist on the horizon itself. It does not have to exist off the horizon. In addition, we find that the limit when the surface gravity of the horizon goes to zero provides a topological constraint. Specifically, the integral of the scalar curvature of the cross sections of the horizon has to be positive when the dominant energy condition is satisfied and the cosmological constant  $\Lambda$  is zero or positive, and in particular rules out the torus topology for supersymmetric isolated horizons (unless  $\Lambda < 0$ ) if and only if the stress-energy tensor  $T_{ab}$  is of the form such that  $T_{ab}\ell^a n^b = 0$  for any two null vectors  $\ell$  and  $n$  with normalization  $\ell_a n^a = -1$  on the horizon.

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## 1 Introduction

Isolated horizons (IHs) were first introduced as an application of the loop quantum gravity (LQG) approach to black hole statistical mechanics [1–3]. It was realized soon after that the framework has a very rich and elegant classical structure [4, 5]. An IH is a null hypersurface at which the intrinsic geometry is held fixed; this generalizes the notion of an event horizon so that the black hole is an object that is in local equilibrium with its (possibly) dynamic environment. Remarkably, the existence of such a surface is sufficient for the zeroth and first laws of black-hole mechanics to be satisfied. However, unlike the older approach that is based on Killing horizons [6–8], the first law of IHs relates quantities that are all defined *at the horizon*.

During the last several years, IHs have been extensively studied. Calculations involving IHs were first done in terms of the (complex or real) self-dual connections and  $SL(2, \mathbb{C})$  soldering forms [2–5], and later were refined in terms of real Lorentz connections and tetrads [9]. The former approach is better suited for quantum applications, while the latter for classical mechanics and geometry. The framework was extensively studied with inclusion of matter fields (dilaton, Yang-Mills etc) [10–12], rotation [13], and a negative cosmological constant in three dimensions [14]. Geometrical issues were extensively studied in [15–17]. Following from earlier work on marginally trapped surfaces [18], IHs were also extended to nonequilibrium black holes known as dynamical [19, 20] and slowly evolving [21] horizons.

IHs have only recently been extended to higher dimensions, notably for vacuum spacetimes [22, 23], asymptotically anti-de Sitter (ADS) spacetimes [24], and Einstein-Gauss-Bonnet spacetimes [25]. Following up on the latter result, it was shown [26] that there is a lower bound on the GB parameter for which the area-increase law can be violated when two black holes merge. The calculation was done for IHs in four dimensions, but presumably the result holds for IHs in higher dimensions as well (although in the latter case the topologies are not as severely restricted as they are in four dimensions, even for Einstein gravity with vanishing cosmological constant [27, 28]).

The purpose of this work is to present a further extension of IHs to supergravity. In particular, we consider an extension of the isolated horizon framework that includes charged and rotating black holes in Einstein-Maxwell (EM) theory with an additional Chern-Simons (CS) term for the electromagnetic field. The present work is therefore a quasilocal generalization of the supersymmetric black holes that were discussed in [29]; analogues of these black holes in asymptotically ADS spacetimes were constructed in [30] and extended to multi-charge solutions in [31]. Here, we include an arbitrary cosmological constant, but do not make any assumptions about

the boundary at infinity.

We consider the phase space of solutions to the equations of motion for the EMCS action

$$S = \frac{1}{2k_D} \int_{\mathcal{M}} d^{D-1}x \sqrt{-g} \left[ R - 2\Lambda - \frac{1}{4}F^2 + \beta \epsilon_{ab_1 \dots b_{D-1}} A^a F^{b_1 b_2} \dots F^{b_{D-2} b_{D-1}} \right]. \quad (1)$$

Here,  $R$  is the scalar curvature of the Lorentzian manifold  $\mathcal{M}$ ,  $F^2 = F_{ab}F^{ab}$  ( $a, b, \dots \in \{0, \dots, D-1\}$ ) with  $F_{ab} = \partial_a A_b - \partial_b A_a$  and  $A_a$  is the electromagnetic vector potential. The constants appearing in the action are the gravitational coupling constant  $k_D = 8\pi G_D$  (with  $G_D$  the  $D$ -dimensional Newton constant), the cosmological constant  $\Lambda$  and the CS parameter  $\beta$ . The last term is a CS term for the electromagnetic field that can be added to the action for odd-dimensional spacetimes.

In Section 2 we rewrite the action (1) in the first-order connection formulation of general relativity, after which we specify the boundary conditions that are imposed onto the inner boundary of  $\mathcal{M}$ ; these capture the notion of a weakly isolated horizon (WIH) that physically corresponds to an isolated black hole in a surrounding spacetime with (possibly dynamical) fields, imply that the horizon electric charge is independent of the choice of horizon cross sections, and leads to the zeroth law of black-hole mechanics. “Weak isolation” is a slightly less restrictive notion than “isolation” in the sense that weak isolation is the very minimum requirement for the zeroth law to follow from the boundary conditions.

In Section 3 we study the mechanics of the WIHs. In particular, we show that the action principle with boundaries is well defined by explicitly showing that the first variation of the surface term vanishes on the horizon. We then find an expression for the symplectic structure by integrating over a spacelike hypersurface the antisymmetrized second variation of the surface term. This allows us to find an expression for the local version of the (equilibrium) first law of black-hole mechanics in dimensions  $D \geq 5$ . Summarizing Section 2 and Section 3, we have the following: **Result 1.** *A charged and rotating WIH  $\Delta \subset \mathcal{M}$  on the phase space of solutions of EMCS theory in  $D$  dimensions satisfies the zeroth and first laws of black-hole mechanics.*

Beginning in Section 4 we restrict our study to the stronger notion of (fully) IHs. The sign of the surface gravity  $\kappa_{(\ell)}$  is well defined for such horizons. The requirement that  $\kappa_{(\ell)} \geq 0$  therefore allowed us to define a parameter that provides a constraint on the topology of the IHs. Specifically, we find that the integral of the scalar curvature of the cross sections of the horizon (in a spacetime with nonnegative cosmological constant) have to be strictly positive if the dominant energy condition is satisfied or otherwise zero iff the horizon is extremal and nonrotating with the stress-energy tensor  $T_{ab}$  of the form such that  $T_{ab}\ell^a n^b = 0$  for any two null vectors  $\ell$  and  $n$  with normalization  $\ell_a n^a = -1$  at the horizon. In the case of electromagnetic fields with or without the CS term, the scalar  $T_{ab}\ell^a n^b$  is the square of the electric

flux crossing the horizon.

In Section 5 we specialize to IHs in four and in five dimensions with vanishing cosmological constant. With  $\beta = 0$  the action (1) is the bosonic part of four-dimensional  $N = 2$  supergravity, and with  $\beta = -2/(3\sqrt{3})$  the action (1) is the bosonic part of five-dimensional  $N = 1$  supergravity. We show that the existence of a Killing spinor on the IH requires that in both four and in five dimensions the induced (normal) connection on the horizon has to vanish. The IH boundary conditions then imply that the surface gravity and rotation one-form of the horizon are zero. This leads to the following:

**Result 2.** *An IH of  $D = 4$  EM theory and of  $D = 5$  EMCS theory with vanishing cosmological constant is supersymmetric only if the surface gravity and rotation one-form are zero.*

The angular momentum of an IH in general contains a contribution from the Maxwell fields. However, this contribution can be shown [13] to be a result of the bulk radiation field integrated to surface terms; the surface term at infinity is zero due to the fall-off conditions imposed on the Maxwell fields. Therefore Result 2 implies that a supersymmetric IH (SIH) is extremal and nonrotating. The Breckenridge-Myers-Peet-Vafa (BMPV) black hole [32] is an example of a distorted horizon with arbitrary rotations in the bulk fields [33]; when the angular momentum vanishes this solution reduces to the extremal Reissner-Nordström (RN) solution [34] in isotropic coordinates. The conclusions of Section 4 imply that the only possible horizon topologies for SIHs are  $S^2$  in four dimensions while  $S^3$  and  $S^1 \times S^2$  in five dimensions. The torus topology is a special case that is allowed only if the square of the electric flux across the horizon is zero.

In Section 6 we conclude with some brief comments about further research that can be done with the formalism developed here.

## 2 Boundary conditions and the zeroth law

### 2.1 First-order action for gravity coupled to electromagnetic fields

For application to IHs, we work in the first-order formulation with the “connection-dynamics” approach. For details we refer the reader to the review [36] and references therein. In this formulation, the configuration space consists of the triple  $(e^I, A^I_J, \mathbf{A})$ ; the coframe  $e^I = e_a^I dx^a$  determines the metric

$$g_{ab} = \eta_{IJ} e_a^I \otimes e_b^J, \quad (2)$$

the gravitational  $SO(D-1, 1)$  connection  $A^I_J = A_a^I{}_J dx^a$  determines the curvature two-form

$$\Omega^I_J = dA^I_J + A^I_K \wedge A^K_J, \quad (3)$$

and the electromagnetic  $U(1)$  connection  $\mathbf{A}$  determines the curvature

$$\mathbf{F} = d\mathbf{A} . \quad (4)$$

In this paper, spacetime indices  $a, b, \dots \in \{0, \dots, D-1\}$  are raised and lowered using the metric tensor  $g_{ab}$ , while internal Lorentz indices  $I, J, \dots \in \{0, \dots, D-1\}$  are raised and lowered using the Minkowski metric  $\eta_{IJ} = \text{diag}(-1, 1, \dots, 1)$ . The curvature  $\Omega$  defines the Riemann tensor  $R^I_{JKL}$  (with the convention of Wald [35]) via

$$\Omega^I_J = \frac{1}{2} R^I_{JKL} e^K \wedge e^L . \quad (5)$$

The Ricci tensor is then  $R_{IJ} = R^K_{IKJ}$ , and the Ricci scalar is  $R = \eta^{IJ} R_{IJ}$ . The gauge covariant derivative  $\mathcal{D}$  acts on generic fields  $\Psi_{IJ}$  such that

$$\mathcal{D}\Psi^I_J = d\Psi^I_J + A^I_K \wedge \Psi^K_J - A^K_J \wedge \Psi^I_K . \quad (6)$$

The coframe defines the  $(D-m)$ -form

$$\Sigma_{I_1 \dots I_m} = \frac{1}{(D-m)!} \epsilon_{I_1 \dots I_m I_{m+1} \dots I_D} e^{I_{m+1}} \wedge \dots \wedge e^{I_D} , \quad (7)$$

where the totally antisymmetric Levi-Civita tensor  $\epsilon_{I_1 \dots I_D}$  is related to the spacetime volume element by

$$\epsilon_{a_1 \dots a_D} = \epsilon_{I_1 \dots I_D} e^{I_1}_{a_1} \dots e^{I_D}_{a_D} . \quad (8)$$

In this configuration space, the action (1) for the theory on the manifold  $(\mathcal{M}, g_{ab})$  (assumed for the moment to have no boundaries) is given by

$$S = \frac{1}{2k_D} \int_{\mathcal{M}} \Sigma_{IJ} \wedge \Omega^{IJ} - 2\Lambda \epsilon - \frac{1}{4} \mathbf{F} \wedge \star \mathbf{F} + \beta \mathbf{A} \wedge \mathbf{F}^{\mathfrak{n}} . \quad (9)$$

Here  $\epsilon = e^0 \wedge \dots \wedge e^{D-1}$  is the spacetime volume element, “ $\star$ ” denotes the Hodge dual, and  $\mathfrak{n} = (D-1)/2$ .

The equations of motion are given by  $\delta S = 0$ , where  $\delta$  is the first variation; i.e. the stationary points of the action. For this configuration space the equations of motion are derived from independently varying the action with respect to the fields  $(e, A, \mathbf{A})$ . To get the equation of motion for the coframe we note the identity

$$\delta \Sigma_{I_1 \dots I_m} = \delta e^M \wedge \Sigma_{I_1 \dots I_m M} . \quad (10)$$

This leads to

$$\Sigma_{IJK} \wedge \Omega^{JK} - 2\Lambda \Sigma_I = \mathcal{T}_I , \quad (11)$$

where  $\mathcal{T}_I$  denotes the electromagnetic stress-energy  $(D-1)$ -form. The equation of motion for the connection  $A$  is

$$\mathcal{D}\Sigma_{IJ} = 0 ; \quad (12)$$

this equation says that the torsion  $T^I = \mathcal{D}e^I$  is zero. The equation of motion for the connection  $\mathbf{A}$  is

$$d \star \mathbf{F} + 4(\mathbf{n} + 1)\beta \mathbf{F}^{\mathbf{n}} = 0 . \quad (13)$$

The second term in this equation is the contribution due to the CS term in the action; when this term is turned off, the equation reduces to the standard Maxwell equation  $d \star \mathbf{F} = 0$ . The equations (11) and (12) are equivalent to the Einstein equations in the metric formulation, with the components of  $\mathcal{T}_I$  identified with the electromagnetic stress-energy tensor.

## 2.2 Boundary conditions

Let us from here on consider the manifold  $(\mathcal{M}, g_{ab})$  to contain boundaries; the conditions that we will impose on the inner boundary will capture the notion of an isolated black hole that is in local equilibrium with its (possibly) dynamic surroundings. For details we refer the reader to [25] and the references listed there. We follow the general recipe that was developed in [9], and later extended to rotating IHs in [13].

First, we give some general comments about the structure of the manifold. Specifically,  $(\mathcal{M}, g_{ab})$  is a  $D$ -dimensional Lorentzian manifold with topology  $R \times M$ , contains a  $(D - 1)$ -dimensional null surface  $\Delta$  as inner boundary (representing the horizon), and is bounded by  $(D - 1)$ -dimensional spacelike surfaces  $M^\pm$  that intersect  $\Delta$  in compact  $(D - 2)$ -spaces  $\mathbb{S}^\pm$  and extend to the boundary at infinity  $\mathcal{B}$ .

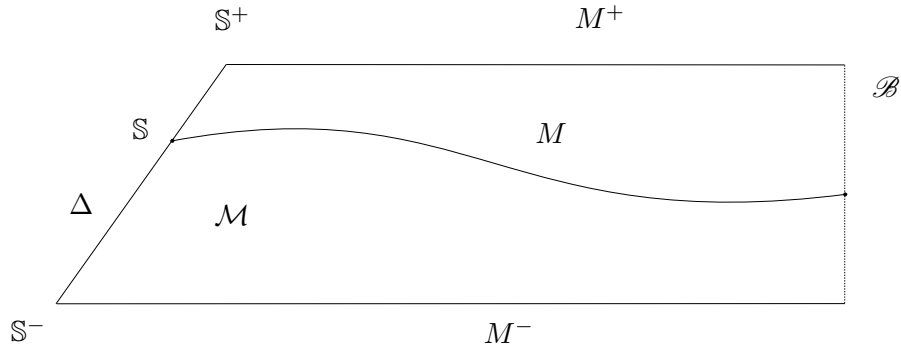


Figure 1: The region of the  $D$ -dimensional spacetime  $\mathcal{M}$  being considered has an internal null boundary  $\Delta$  representing the event horizon, and is bounded by  $(D - 1)$ -dimensional spacelike surfaces  $M^\pm$  which intersect  $\Delta$  in compact  $(D - 2)$ -spaces  $\mathbb{S}^\pm$  and extend to the boundary at infinity  $\mathcal{B}$ . Here,  $M$  is a  $(D - 1)$ -dimensional partial Cauchy surface.

The outer boundary  $\mathcal{B}$  is some arbitrary  $(D - 1)$ -dimensional surface, and is loosely referred to as the “boundary at infinity”. In this paper, as in [25], we

consider the purely quasilocal case and neglect any subtleties that are associated with the outer boundary. Including this contribution in the phase space amounts to imposing fall-off conditions on the fields for fixed  $\Lambda$  (e.g. asymptotically flat [9] or asymptotically ADS [24]) as they approach  $\mathcal{B}$ .

Fixing the geometric structures and matter fields on  $\Delta$  captures the notion of a non-expanding horizon [9]:

**Definition I.** *A non-expanding horizon  $(\Delta, q_{ab}, \ell_a)$  is a  $(D-1)$ -dimensional null hypersurface  $\Delta$  with topology  $R \times \mathbb{S}^{D-2}$  together with a degenerate metric  $q_{ab}$  of signature  $0 + \dots +$  (with  $D-2$  nondegenerate spatial directions) and a null normal  $\ell_a$  such that: (a) the expansion  $\theta_{(\ell)}$  of  $\ell_a$  vanishes on  $\Delta$ ; (b) the field equations hold on  $\Delta$ ; and (c) the stress-energy tensor is such that the vector  $-T^a_b \ell^b$  is a future-directed and causal vector.*

This is the standard definition for a spacetime that satisfies the Einstein field equations. Conditions (a) and (c), together with the Frobenius theorem (which implies that the rotation tensor  $\omega_{ab}$  vanishes on  $\Delta$ ) and the Raychaudhuri equation, then implies that the shear tensor  $\sigma_{ab}$  vanishes on  $\Delta$ . The vanishing of all these quantities implies that

$$\nabla_{\underline{a}} \ell_b \approx \omega_a \ell_b, \quad (14)$$

with “ $\approx$ ” denoting equality restricted to  $\Delta \subset \mathcal{M}$  and the underarrow indicating pull-back to  $\Delta$ . Thus the one-form  $\omega$  is the natural connection (in the normal bundle) induced on the horizon.

A further consequence of the conditions (a) and (c) together with the Raychaudhuri equation is that the EM stress-energy tensor (which is the usual one for electromagnetism because variation of the CS term with respect to  $g_{ab}$  is zero) satisfies the constraint [9]

$$T_{ab} \ell^a \ell^b \approx 0. \quad (15)$$

and therefore that

$$\underline{\ell} \lrcorner \underline{\mathbf{F}} = 0. \quad (16)$$

With the Maxwell-CS equations and the Bianchi identity, it then follows that

$$\mathcal{L}_{\underline{\ell}} \underline{\mathbf{F}} \approx \underline{\ell} \lrcorner d\underline{\mathbf{F}} + d(\underline{\ell} \lrcorner \underline{\mathbf{F}}) = 0. \quad (17)$$

One of the physical consequences of this restriction is that the total electric charge is independent of the choice of cross sections  $\mathbb{S}^{D-2}$ . This is discussed in detail in [5].

### 2.3 Zeroth law

The “time-independence” of  $\omega$  and  $\underline{\mathbf{A}}$  on  $\Delta$  captures the notion of a WIH [9]:

**Definition II.** A *WIH*  $(\Delta, q_{ab}, [\ell])$  is a non-expanding horizon  $\Delta$  together with an equivalence class of null normals  $[\ell]$  such that  $\mathcal{L}_\ell \omega_a = 0$  and  $\mathcal{L}_\ell \underline{\mathbf{A}} = 0$  for all  $\ell \in [\ell]$  (where  $\ell' \sim \ell$  if  $\ell' = c\ell$  for some constant  $c$ ).

The above is not a physical restriction. It is a restriction on the rescaling freedom of  $\ell$  and the gauge freedom of  $\mathbf{A}$ . Now, the surface gravity and electromagnetic scalar potential are defined respectively as  $\kappa_{(\ell)} = \ell^a \omega_a$  and  $\Phi_{(\ell)} = -\ell^a A_a$ . In general, these quantities will change under rescalings of the normal. However, the conditions  $\mathcal{L}_\ell \omega_a = 0$  and  $\mathcal{L}_\ell \mathbf{A} = 0$  are sufficient to ensure that  $d\kappa_{(\ell)} = d(\ell^a \omega_a) = 0$  and  $d\Phi_{(\ell)} = d(\ell^a A_a) = 0$  [9]. The zeroth law therefore follows from the boundary conditions and is independent of the functional content of the action (and in particular of the material content of the stress-energy tensor that describes the coupled matter fields).

### 3 Mechanics and the first law

#### 3.1 Variation of the boundary term

Let us now look at the variation of the action (9). Denoting the triple  $(e, A, \mathbf{A})$  collectively as a generic field variable  $\Psi$ , the first variation gives

$$\delta S = \frac{1}{2k_D} \int_{\mathcal{M}} E[\Psi] \delta \Psi - \frac{1}{2k_D} \int_{\partial \mathcal{M}} J[\Psi, \delta \Psi] . \quad (18)$$

Here  $E[\Psi] = 0$  symbolically denotes the equations of motion and

$$J[\Psi, \delta \Psi] = \Sigma_{IJ} \wedge \delta A^{IJ} - \Phi \wedge \delta \mathbf{A} \quad (19)$$

is the surface term with  $(D-2)$ -form

$$\Phi = \star \mathbf{F} + 4(\mathfrak{n} + 1) \beta \mathbf{A} \wedge \mathbf{F}^{\mathfrak{n}-1} . \quad (20)$$

If the integral of  $J$  on the boundary  $\partial \mathcal{M}$  vanishes then the action principle is said to be differentiable. We must show that this is the case. Because the fields are held fixed at  $M^\pm$  and at  $\mathcal{B}$ ,  $J$  vanishes there. Therefore it suffices to show that  $J$  vanishes at the inner boundary  $\Delta$ . To show that this is true we need to find an expression for  $J$  in terms of  $\Sigma$ ,  $A$  and  $\mathbf{A}$  pulled back to  $\Delta$ . As for the gravitational variables, this is accomplished by employing a higher-dimensional analogue of the Newman-Penrose (NP) formalism [37]. In particular, we fix an internal basis consisting of the (null) pair  $(\ell, n)$  and  $D-2$  spacelike vectors  $\vartheta_{(i)}$  ( $i \in \{2, \dots, D-1\}$ ) such that

$$e_0 = \ell, \quad e_1 = n, \quad e_i = \vartheta_{(i)}, \quad (21)$$

together with the conditions

$$\ell \cdot n = -1, \quad \ell \cdot \ell = n \cdot n = \ell \cdot \vartheta_{(i)} = n \cdot \vartheta_{(i)} = 0, \quad \vartheta_{(i)} \cdot \vartheta_{(j)} = \delta_{ij} . \quad (22)$$



In the following we also apply the summation convention over repeated spacelike indices ( $i, j, k$  etc.). As these are Euclidean indices their position (up or down) will be adjusted according to the dictates of notational convenience.

To find the pull-back to  $\Delta$  of  $\Sigma$ , we use the decomposition

$$e_{\underline{a}}^I \approx -\ell^I n_a + \vartheta_{(i)}^I \vartheta_a^{(i)}, \quad (23)$$

whence the  $(D-2)$ -form

$$\begin{aligned} \underline{\Sigma}_{IJ} \approx & -\frac{1}{(D-3)!} \epsilon_{IJA_1 \dots A_{D-2}} \ell^{A_1} \vartheta_{(i_1)}^{A_2} \dots \vartheta_{(i_{D-3})}^{A_{D-2}} \left( n \wedge \vartheta^{(i_1)} \wedge \dots \wedge \vartheta^{(i_{D-3})} \right) \\ & + \frac{1}{(D-2)!} \epsilon_{IJA_1 \dots A_{D-2}} \vartheta_{(i_1)}^{A_1} \dots \vartheta_{(i_{D-2})}^{A_{D-2}} \left( \vartheta^{(i_1)} \wedge \dots \wedge \vartheta^{(i_{D-2})} \right). \end{aligned} \quad (24)$$

To find the pull-back of  $A$  we first note that [9]

$$\nabla_{\underline{a}} \ell_I \approx \omega_a \ell_I. \quad (25)$$

Then, taking the covariant derivative of  $\ell$  acting on internal indices gives

$$\nabla_a \ell_I = \partial_a \ell_I + A_{aIJ} \ell^J, \quad (26)$$

with  $\partial$  representing a flat derivative operator that is compatible with the internal coframe on  $\Delta$ . Thus  $\partial_a \ell_I \approx 0$  and

$$\nabla_{\underline{a}} \ell_I \approx A_{\underline{a}IJ} \ell^J. \quad (27)$$

Putting this together with (25) we have that

$$A_{\underline{a}IJ} \ell^J \approx \omega_a \ell_I, \quad (28)$$

and this implies that the pull-back of  $A$  to the horizon is of the form

$$A_{\underline{a}}^{IJ} \approx -2\ell^{[I} n^{J]} \omega_a + a_a^{(i)} \ell^{[I} \vartheta_{(i)}^{J]} + b_a^{(ij)} \vartheta_{(i)}^{[I} \vartheta_{(j)}^{J]}, \quad (29)$$

where the  $a_a^{(i)}$  and  $b_a^{(ij)}$  are one-forms in the cotangent space  $T^*(\Delta)$ . It follows that the variation of (29) is

$$\delta A_{\underline{a}}^{IJ} \approx -2\ell^{[I} n^{J]} \delta \omega_a + \delta a_a^{(i)} \ell^{[I} \vartheta_{(i)}^{J]} + \delta b_a^{(ij)} \vartheta_{(i)}^{[I} \vartheta_{(j)}^{J]}. \quad (30)$$

Finally, by direct calculation, it can be shown that the gravitational part  $J_{\text{Grav}}$  of the surface term (19) reduces to

$$J_{\text{Grav}}[\Psi, \delta\Psi] \approx \tilde{\epsilon} \wedge \delta\omega. \quad (31)$$

Here,  $\tilde{\epsilon} = \vartheta^{(1)} \wedge \dots \wedge \vartheta^{(D-2)}$  is the area element of the cross sections  $\mathbb{S}^{D-2}$  of the horizon.

Now we make use of the fact that, because  $\ell$  is normal to the surface, its variation will also be normal to the surface. That is,  $\delta\ell \propto \ell$  for some  $\ell$  fixed in  $[\ell]$ . This

together with  $\mathcal{L}_\ell \omega = 0$  then implies that  $\mathcal{L}_\ell \delta \omega = 0$ . However,  $\omega$  is held fixed on  $M^\pm$  which means that  $\delta \omega = 0$  on the initial and final cross-sections of  $\Delta$  (i.e. on  $M^- \cap \Delta$  and on  $M^+ \cap \Delta$ ), and because  $\delta \omega$  is Lie dragged on  $\Delta$  it follows that  $J_{\text{Grav}} \approx 0$ . The same argument also holds for the electromagnetic part  $J_{\text{EM}}$  of the surface term (19). In particular, because the electromagnetic field is in a gauge adapted to the horizon,  $\mathcal{L}_\ell \underline{\mathbf{A}} = 0$ , and with  $\delta \ell \propto \ell$  we also have that  $\mathcal{L}_\ell \delta \underline{\mathbf{A}} = 0$ . This is sufficient to show that  $J_{\text{EM}} \approx 0$  as well. Therefore the surface term  $J|_{\partial \mathcal{M}} = 0$  for the Einstein-Maxwell theory with electromagnetic CS term, and we conclude that the equations of motion  $E[\Psi] = 0$  follow from the action principle  $\delta S = 0$ .

### 3.2 Covariant phase space

The derivation of the first law involves two steps. First we need to find the symplectic structure on the covariant phase space  $\Gamma$  consisting of solutions  $(e, A, \mathbf{A})$  to the field equations (11), (12) and (13) on  $\mathcal{M}$ . Once we have a suitable (closed and conserved) symplectic two-form, we then need to specify an evolution vector field  $\xi^a$ . In this section we derive the symplectic two-form. In the next section we will specify the evolution vector field which will also serve to introduce an appropriate notion of horizon angular momentum.

The antisymmetrized second variation of the surface term gives the symplectic current, and integrating over a spacelike hypersurface  $M$  gives the symplectic structure  $\Omega \equiv \Omega(\delta_1, \delta_2)$  (with the choice of  $M$  being arbitrary). Following [9], we find that the second variation of the surface term (19) gives

$$J[\Psi, \delta_1 \Psi, \delta_2 \Psi] = \delta_1 \Sigma_{IJ} \wedge \delta_2 A^{IJ} - \delta_2 \Sigma_{IJ} \wedge \delta_1 A^{IJ} - \delta_1 \Phi \wedge \delta_2 \mathbf{A} - \delta_2 \Phi \wedge \delta_1 \mathbf{A} . \quad (32)$$

Whence integrating over  $M$  defines the *bulk* symplectic structure

$$\Omega_{\text{bulk}} = \frac{1}{2k_D} \int_M [\delta_1 \Sigma_{IJ} \wedge \delta_2 A^{IJ} - \delta_2 \Sigma_{IJ} \wedge \delta_1 A^{IJ} - \delta_1 \Phi \wedge \delta_2 \mathbf{A} + \delta_2 \Phi \wedge \delta_1 \mathbf{A}] . \quad (33)$$

We also need to find the pull-back of  $J$  to  $\Delta$  and add the integral of this term to  $\Omega_{\text{bulk}}$  so that the resulting symplectic structure on  $\Gamma$  is conserved. If we define potentials  $\psi$  and  $\chi$  for the surface gravity  $\kappa_{(\ell)}$  and electric potential  $\Phi_{(\ell)}$  such that

$$\mathcal{L}_\ell \psi \approx \ell \lrcorner \omega = \kappa_{(\ell)} \quad \text{and} \quad \mathcal{L}_\ell \chi \approx \ell \lrcorner \mathbf{A} = -\Phi_{(\ell)} , \quad (34)$$

then the pullback to  $\Delta$  of the symplectic structure will be a total derivative; using the Stokes theorem this term becomes an integral over the cross sections  $\mathbb{S}^{D-2}$  of  $\Delta$ . Hence the full symplectic structure is given by

$$\begin{aligned} \Omega &= \frac{1}{2k_D} \int_M [\delta_1 \Sigma_{IJ} \wedge \delta_2 A^{IJ} - \delta_2 \Sigma_{IJ} \wedge \delta_1 A^{IJ} - \delta_1 \Phi \wedge \delta_2 \mathbf{A} + \delta_2 \Phi \wedge \delta_1 \mathbf{A}] \\ &\quad + \frac{1}{k_D} \oint_{\mathbb{S}^{D-2}} [\delta_1 \tilde{\epsilon} \wedge \delta_2 \psi - \delta_2 \tilde{\epsilon} \wedge \delta_1 \psi + \delta_1 \Phi \wedge \delta_2 \chi - \delta_2 \Phi \wedge \delta_1 \chi] . \end{aligned} \quad (35)$$

### 3.3 Angular momentum and the first law

In  $D$  dimensions, there are  $\tilde{n} = \lfloor (D-1)/2 \rfloor$  rotation parameters given by the Casimirs of the rotation group  $SO(D-1)$ . Here, “ $\lfloor (D-1)/2 \rfloor$ ” denotes the integer value of  $\mathbf{n}$ . For a multidimensional WIH rotating with angular velocities  $\Omega_\iota$  ( $\iota = 1, \dots, \tilde{n}$ ), a suitable evolution vector field on the covariant phase space is given by [13, 24]

$$\xi^a = k\ell^a + \sum_{\iota=1}^{\tilde{n}} \Omega_\iota \phi_\iota^a . \quad (36)$$

Here,  $k$  is a constant on  $\Delta$  and  $\phi_\iota^a$  are spacelike rotational vector fields that satisfy

$$\mathcal{L}_\phi q_{ab} = 0, \quad \mathcal{L}_\phi \ell_a = 0, \quad \mathcal{L}_\phi \omega_a = 0, \quad \mathcal{L}_\phi \underline{\mathbf{A}} = 0, \quad \mathcal{L}_\phi \underline{\mathbf{F}} = 0 . \quad (37)$$

The vector field  $\xi$  is similar to the linear combination  $\zeta = t + \sum_\iota \Omega_\iota m_\iota$  (with  $t$  a timelike Killing vector and  $m_\iota$  spacelike Killing vectors) for globally stationary spacetimes which on a Killing horizon is null. By contrast, we note that  $\xi$  is null on  $\Delta$  only when all angular momenta are zero; in general  $\xi$  is *spacelike*.

Moving on, the first law now follows directly from evaluating the symplectic structure at  $(\delta, \delta_\xi)$ . See e.g. [24] for details. This gives two surface terms: one at infinity (which is identified with the ADM energy), and one at the horizon. We find that the surface term at the horizon is given by

$$\Omega|_\Delta = \frac{\kappa_{(k\ell)}}{k_D} \delta \oint_{\mathbb{S}^{D-2}} \tilde{\epsilon} + \frac{\Phi_{(k\ell)}}{k_D} \delta \oint_{\mathbb{S}^{D-2}} \Phi + \sum_{\iota=1}^{\tilde{n}} \frac{\Omega_\iota}{k_D} \delta \oint_{\mathbb{S}^{D-2}} [(\phi_\iota \lrcorner \omega) \tilde{\epsilon} + (\phi_\iota \lrcorner \mathbf{A}) \Phi] , \quad (38)$$

where we used  $\kappa_{(k\ell)} = \mathcal{L}_{k\ell} \psi = k\ell \lrcorner \omega$  and  $\Phi_{(k\ell)} = \mathcal{L}_{k\ell} \chi = k\ell \lrcorner \omega$ . These potentials are constant for any given horizon, but in general vary across the phase space from one point to another. This implies that (38) is *not* in general a total variation. However, an appropriate normalization of the vector  $k\ell^a$  can be chosen so that (38) is a total variation. The necessary and sufficient condition for the system to be Hamiltonian is that there exists a function  $\mathcal{E}$  such that  $\Omega|_\Delta(\delta, \delta_\xi) = \delta \mathcal{E}$ . Given that such a function does exist, we find that

$$\delta \mathcal{E} = \frac{\kappa_{(t)}}{k_D} \delta \oint_{\mathbb{S}^{D-2}} \tilde{\epsilon} + \frac{\Phi_{(k\ell)}}{k_D} \delta \oint_{\mathbb{S}^{D-2}} \Phi + \sum_{\iota=1}^{\tilde{n}} \frac{\Omega_\iota}{k_D} \delta \oint_{\mathbb{S}^{D-2}} [(\phi_\iota \lrcorner \omega) \tilde{\epsilon} + (\phi_\iota \lrcorner \mathbf{A}) \Phi] . \quad (39)$$

This is the first law of black-hole mechanics. For a quasi-static process, the standard form of the first law of thermodynamics is

$$\delta \mathcal{E} = T_{(k\ell)} \delta \mathcal{S} + \Phi_{(k\ell)} \delta \mathcal{Q} + \sum_{\iota=1}^{\tilde{n}} \Omega_\iota \delta \mathcal{J}_\iota ; \quad (40)$$

comparing this to (39), and with the identification of the Hawking temperature  $T_{(k\ell)} = \kappa_{(k\ell)}/(2\pi)$ , we find that the entropy, electric charge and angular momenta

of the isolated horizon are:

$$\mathcal{S} = \frac{1}{4G_D} \oint_{\mathbb{S}^{D-2}} \tilde{\epsilon} \quad (41)$$

$$\mathcal{Q} = \frac{1}{8\pi G_D} \oint_{\mathbb{S}^{D-2}} \star \mathbf{F} + 4(n+1)\beta \mathbf{A} \wedge \mathbf{F}^{n-1} \quad (42)$$

$$\mathcal{J}_\ell = \frac{1}{8\pi G_D} \oint_{\mathbb{S}^{D-2}} [(\phi_\ell \lrcorner \omega) \tilde{\epsilon} + (\phi_\ell \lrcorner \mathbf{A}) \Phi] . \quad (43)$$

Therefore WIHs in  $D$ -dimensional EMCS theory satisfy the first law (and the zeroth law) of black-hole mechanics. This is in agreement with [29], but with a very important difference. Here, all the quantities appearing in the first law are defined at the horizon; no reference was made to the boundary at infinity.

**Remark.** The expression (43) implies that the horizon angular momentum contains contributions from both gravitational *and* electromagnetic fields, here referred to as  $\mathcal{J}_{\text{Grav}}$  and  $\mathcal{J}_{\text{EM}}$ . This is in contrast to the standard angular momentum expressions at infinity, such as the Komar expression. One can show [13, 24] that  $\mathcal{J}_{\text{Grav}}$  is equivalent to the (quasilocal) Komar integral

$$\mathcal{J}_K = -\frac{1}{8\pi G_D} \oint_{\mathbb{S}^{D-2}} \star d\phi, \quad (44)$$

and this matches the expression for Killing horizons at infinity.

It would appear that if  $\omega = 0$  then there is still a nonzero contribution to (43) from  $\mathcal{J}_{\text{EM}}$ . However, it can be shown [13, 24] that if  $\phi$  is the restriction to  $\Delta$  of a *global* rotational Killing field  $\varphi$  contained in  $\mathcal{M}$ , then  $\mathcal{J}_{\text{EM}}$  is actually the angular momentum of the electromagnetic radiation in the bulk. What happens is that the bulk integral can be written as the sum of a surface term at  $\Delta$  and a surface term at  $\mathcal{B}$ , and the latter vanishes due to the fall-off conditions on the fields. Therefore we say that a nonrotating WIH is one for which  $\omega = 0$ .

## 4 A topological constraint from extremality

One of the properties of an extremal black hole is that its surface gravity is zero. Another property is that its horizons are degenerate: the inner and outer horizons coincide. As a result, an extremal black hole is one for which there are no trapped surfaces “just inside” the horizon. This property was recently used [38] to define an extremality condition for quasilocal horizons. We note here the evolution equation for the expansion of the null normal  $n^a$  [38]:

$$\mathcal{L}_\ell \theta_{(n)} + \kappa_{(n)} + \frac{1}{2} \mathcal{R} = d_a \tilde{\omega}^a + \|\tilde{\omega}\|^2 + (T_{ab} - \Lambda g_{ab}) \ell^a n^b . \quad (45)$$

Here,  $\mathcal{R}$  is the scalar curvature of  $\mathbb{S}^{D-2}$ ,  $d_a$  is the covariant derivative operator that is compatible with the metric  $\tilde{q}_{ab} = g_{ab} + \ell_a n_b + \ell_b n_a$ , and  $\|\tilde{\omega}\|^2 = \tilde{\omega}_a \tilde{\omega}^a$  where

$$\tilde{\omega}_a = \tilde{q}_a{}^b \omega_b = \omega_a + \kappa_{(\ell)} n_a \quad (46)$$

is the projection of  $\omega$  onto  $\mathbb{S}^{D-2}$ .  $\tilde{\omega}$  is referred to as the rotation one-form.

Our desire is to apply the expression (45) to *black holes*, and in order to do this we need to impose some restrictions on the WIHs. Henceforth we will restrict our attention to (fully) IHs. These are WIHs for which there is a scaling of the null normals for which the commutator  $[\mathcal{L}_\ell, \mathcal{D}] = 0$  (with  $\mathcal{D}$  the intrinsic covariant derivative on the horizon). Physically these IHs have time-invariant extrinsic (as well as intrinsic) geometries and, up to a free constant, a preferred scaling of the null normals. The condition for full isolation, however, still does not guarantee that we are dealing with black holes. There are many examples of IHs which do not describe black holes. An example is a vanishing scalar invariant (VSI) spacetime for which it has been shown that no trapped surfaces exist [17, 40]. Thus we restrict our attention to IHs for which there do exist trapped surfaces, in which case then  $\theta_{(n)} < 0$ . Then the sign of the surface gravity is well defined and a horizon is sub-extremal if  $\kappa > 0$  and extremal (with degenerate horizons) if  $\kappa = 0$ . Further,  $\mathcal{L}_\ell \theta_{(n)} = 0$  and combining this with the fact that the inward expansion  $\theta_{(n)}$  should always be less than zero, an integration of (45) gives

$$\eta \equiv \oint_{\mathbb{S}^{D-2}} \tilde{\epsilon} (T_{ab} \ell^a n^b + \Lambda + \|\tilde{\omega}\|^2) - \frac{1}{2} \oint_{\mathbb{S}^{D-2}} \tilde{\epsilon} \mathcal{R} \leq 0. \quad (47)$$

(Here we used  $-\Lambda g_{ab} \ell^a n^b = \Lambda$  to simplify the cosmological term, and the fact that  $\oint_{\mathbb{S}^{D-2}} \tilde{\epsilon} d_a \tilde{\omega}^a = 0$ .) This inequality provides an alternative characterization of extremal IHs: if  $\eta < 0$  ( $\kappa > 0$  and  $\theta_{(n)} < 0$ ) then  $\Delta$  is nonextremal, and if  $\eta = 0$  ( $\kappa = 0$ ) then  $\Delta$  is extremal. However, this inequality also provides a topological constraint on the cross sections of  $\Delta$ . To see this, rewrite (47) so that

$$\oint_{\mathbb{S}^{D-2}} \tilde{\epsilon} (\mathcal{R} - 2\Lambda) \geq 2 \oint_{\mathbb{S}^{D-2}} \tilde{\epsilon} (T_{ab} \ell^a n^b + \|\tilde{\omega}\|^2). \quad (48)$$

Now, observe that the dominant energy condition requires that  $T_{ab} \ell^a n^b \geq 0$ . In addition,  $\|\tilde{\omega}\|^2$  is manifestly non-negative. The inequality (48) therefore restricts the topology of the cross sections of the horizon. The condition (48) is the same as the one that was found in four dimensions for marginally trapped surfaces [18], nonexpanding horizons [17] and dynamical horizons [20, 39].

For nonextremal horizons,  $\eta < 0$ , and the constraint (48) splits into two possibilities, depending on the nature of the cosmological constant:

- $\Lambda \geq 0$ . The integral of the scalar curvature is strictly positive. In four dimensions the GB theorem says that  $\oint_{\mathbb{S}^2} \tilde{\epsilon} \mathcal{R} = 8\pi(1 - g)$ , with  $g$  the genus of the surface  $\mathbb{S}^2$ . In this case  $\eta < 0$  implies that  $g = 0$  and hence the only possibility is that the cross sections are two-spheres  $S^2$ . In five dimensions  $\eta < 0$  implies that the cross sections are of positive Yamabe type; this implies that topologically  $\mathbb{S}^3$  can only be a finite connected sum of the three-sphere  $S^3$  or of the ring  $S^1 \times S^2$  [28, 41]. Both these topologies have been realized

and the corresponding solutions, for example the Myers-Perry black hole [34] and the Emparan-Reall black ring [42], are well known.

- $\Lambda < 0$ . The integral of the scalar curvature can have either sign, or even vanish, and the inequality will always be satisfied. The only restriction is that

$$\oint_{\mathbb{S}^{D-2}} \tilde{\epsilon}(\mathcal{R} + 2|\Lambda|) \geq 2 \oint_{\mathbb{S}^{D-2}} \tilde{\epsilon}(T_{ab}\ell^a n^b + \|\tilde{\omega}\|^2). \quad (49)$$

There is no constraint on the topology of  $\mathbb{S}^{D-2}$ . Owing to this special property, and to the realization that black holes can be constructed with suitable identifications of points in ADS spacetime, many such black holes have been found with exotic topologies in  $D \geq 3$  dimensions. See e.g. [43–47].

For extremal horizons,  $\eta = 0$ , and the constraint (48) becomes an equality. In this case the same restrictions apply to  $\oint_{\mathbb{S}^{D-2}} \tilde{\epsilon}\mathcal{R}$  as for nonextremal horizons. However, there is also a special case that occurs:

$$\oint_{\mathbb{S}^{D-2}} \tilde{\epsilon}\mathcal{R} = 0 \quad (50)$$

for an extremal and nonrotating ( $\omega = \tilde{\omega} = 0$ ) horizon when the scalar  $T_{ab}\ell^a n^b$  vanishes on the horizon. This case corresponds to the torus topology  $T^{D-2}$ .

## 5 Supersymmetric isolated horizons

### 5.1 Killing spinors in $D = 4$

So far in this paper we have discussed physical aspects of WIHs and IHs in arbitrary dimensions. However, the main purpose here is to find the conditions for supersymmetry. Thus we will now examine the special cases of IHs in  $D = 4$  and  $D = 5$  dimensions. From here on we will also restrict our covariant phase space to solutions for which  $\Lambda = 0$ .

We will first consider the four-dimensional action with  $\beta = 0$ . This is just EM theory, which can be embedded into  $N = 2$  supergravity; the (extremal) RN black hole, which is a solution to the EM theory, is also a solution to the  $N = 2$  supergravity with the fermion fields set to zero. As was shown in [48], the condition for a supersymmetric black hole in four dimensions to have positive energy is that  $M_\infty = |Q_\infty|$  which is the extremality condition for the RN black hole relating the mass  $M_\infty$  and charge  $Q_\infty$  at infinity. This is also the saturated Bogomol’ny-Prasad-Sommerfeld (BPS) inequality.

In general, solutions to the supergravity equations of motion have to be invariant under the supersymmetry transformations of the fields. Black holes in particular are solutions to the bosonic equations of motion, which means that the fermion fields vanish. This implies that the transformations of the bosonic fields  $e$  and  $\mathbf{A}$

vanish when evaluated on the solutions, and therefore the only condition for supersymmetry comes from the fermion variation: the necessary and sufficient condition for supersymmetry with vanishing fermion fields is that there exists a Killing spinor  $\psi_{AA'} = (\alpha_A, \beta_{A'})$  ( $A, B, \dots \in \{1, 2\}$  and  $A', B', \dots \in \{1, 2\}$ ) such that [49, 50]

$$\nabla_{AA'}\alpha_B + \sqrt{2}\phi_{AB}\beta_{A'} = 0 \quad (51)$$

$$\nabla_{AA'}\beta_{B'} - \sqrt{2}\bar{\phi}_{A'B'}\alpha_A = 0. \quad (52)$$

Here,  $\phi_{AB}$  is the Maxwell spinor and  $\bar{\phi}_{A'B'}$  is its complex conjugate. These are related to the field strength via

$$\mathbf{F} = \phi_{AB}\epsilon_{A'B'} + \bar{\phi}_{A'B'}\epsilon_{AB}; \quad (53)$$

the spinor symplectic structure is defined such that  $\epsilon^{12} = -\epsilon^{21} = 1$ . Using the spinors  $\alpha$  and  $\beta$  we can define the following set of null vectors:

$$\ell_{AA'} = \alpha_A\bar{\alpha}_{A'}, \quad n_{AA'} = \bar{\beta}_A\beta_{A'}, \quad \vartheta_{AA'} = \alpha_A\beta_{A'}, \quad \bar{\vartheta}_{AA'} = \bar{\beta}_A\bar{\alpha}_{A'}. \quad (54)$$

It can be shown that the vector

$$K_{AA'} \equiv \ell_{AA'} + n_{AA'} \quad (55)$$

is a Killing vector; the norm of this vector is given by

$$\|K\| = 2V\bar{V}, \quad (56)$$

where we defined the scalar  $V = \alpha_A\bar{\beta}^A$ . It follows that  $K_{AA'}$  can be either timelike (referred to as nondegenerate) when  $V \neq 0$  or null (referred to as degenerate) when  $V = 0$ .

For IHs, the case of interest is the one for which the Killing spinor is null. This is a particularly special case because  $V = \alpha_A\bar{\beta}^A = 0$  implies that

$$\bar{\beta}^A = \mathcal{K}\alpha^A \quad (57)$$

for some function  $\mathcal{K}$ . Putting this into the conditions (51) and (52) allows one to find an expression for the covariant derivative in terms of the spinors [49, 50]:

$$\nabla_{AA'}\alpha_B = -\sqrt{2}\bar{\mathcal{K}}\phi\alpha_A\alpha_B\bar{\alpha}_{A'}. \quad (58)$$

Here,  $\phi$  is a function defined by  $\phi_{AB} = \phi\alpha_A\alpha_B$ . Let us use this form of the covariant derivative to find the covariant derivative of the null normal  $\ell$  of an IH. We find that

$$\nabla_a\ell_b = -\sqrt{2}(\bar{\mathcal{K}}\phi + \mathcal{K}\bar{\phi})\ell_a\ell_b. \quad (59)$$

This immediately implies that

$$\nabla_{\underline{a}}\ell_b \approx 0, \quad (60)$$

and with (14) it follows that  $\omega = 0$ . Therefore the existence of a Killing spinor on an IH implies that the connection one-form has to be zero by the boundary conditions. We emphasize, however, that in this calculation the Killing spinor only has to exist on the horizon itself. It does not have to exist off the horizon. We will discuss the implications of this constraint in Subsection 5.3, after we look at the five-dimensional EMCS theory.

## 5.2 Killing spinors in $D = 5$

We will now consider the five-dimensional action. With  $D = 5$  and  $\beta = -2/(3\sqrt{3})$  the action (9) is the bosonic sector of  $N = 1$  supergravity. As in the four-dimensional EM theory, solutions to the bosonic equations of motion require the existence of a Killing spinor to ensure that supersymmetry is preserved. For black holes, the positive energy theorem together with this requirement imply that the mass and charge are constrained such that  $M_\infty = (\sqrt{3}/2)|Q_\infty|$  [51]. As can be verified, this equality is satisfied by the (5D) extremal RN black hole [34], the BMPV black hole [32] and the Elvang-Empanan-Mateos-Reall (EEMR) black ring [52].

The strategy for finding supersymmetric solutions to the bosonic equations of motion based on Killing spinors is essentially the same in five dimensions as it is in four dimensions. Using the field equations, one determines the constraints on various bosonic bilinears that are constructed from the Killing spinor, which are then solved to derive the spacetime metrics and their associated Maxwell fields. This is the strategy that was pioneered by Tod for supergravity in four dimensions, and later extended to five dimensions by Gauntlett *et al* [53].

The situation in  $D \geq 5$  dimensions is more complicated from an algebraic point of view because, unlike the case of supergravity in four dimensions, we cannot use the simple Newman-Penrose basis as we did in the preceding subsection. An additional problem arises specifically in five dimensions – spinors satisfying certain reality conditions cannot be consistently defined unless they come in pairs and are equipped with a symplectic structure. For details we refer the interested reader to the excellent Les Houches lectures by Van Nieuwenhuizen [54].

In what follows, we shall employ the conventions of [53] and consider (commuting) symplectic Majorana spinors  $\epsilon^\alpha$  ( $\alpha, \beta, \dots \in \{1, 2\}$ ; we keep in mind that unlike the spinor indices  $A, A'$  in the preceding subsection, these spinor indices denote spinors rather than the components of spinors) together with a set of gamma matrices  $\Gamma^I$  that satisfy the anticommutation rule

$$\Gamma^I \Gamma^J + \Gamma^J \Gamma^I = 2\eta^{IJ} \quad (61)$$

and the antisymmetry product

$$\Gamma_{IJKLM} = \epsilon_{IJKLM} . \quad (62)$$

In general,  $\Gamma_{I_1 \dots I_D}$  denotes the antisymmetrized product of  $D$  gamma matrices.



The spinors  $\epsilon^\alpha$  satisfy the reality condition

$$\bar{\epsilon}^\alpha \equiv (\epsilon^\alpha)^\dagger \Gamma_0 = (\epsilon^\alpha)^T \mathcal{C}; \quad (63)$$

$\dagger$  denotes Hermitian conjugation,  $T$  denotes matrix transpose, and  $\mathcal{C}$  is the charge conjugation operator satisfying

$$\mathcal{C}(\Gamma_I)^T \mathcal{C}^{-1} = \Gamma_I. \quad (64)$$

Spinor indices are raised and lowered using the symplectic structure  $\epsilon_{\alpha\beta}$  which is defined such that  $\epsilon_{12} = \epsilon^{12} = +1$ .

Just as for bosonic fields in four dimensions, the necessary and sufficient condition for supersymmetry with vanishing fermion fields is that there exists a Killing spinor. In the case of five-dimensional supergravity,  $\epsilon^\alpha$  are Killing spinors iff

$$\left[ \nabla_I + \frac{1}{4\sqrt{3}} (\Gamma_I^{JK} - 4\delta_I^J \Gamma^K) F_{JK} \right] \epsilon^\alpha = 0. \quad (65)$$

Using  $\epsilon^\alpha$  we can construct bosonic bilinears  $f$ ,  $V^I$  and  $\Phi^{\alpha\beta} = \Phi^{(\alpha\beta)}$  such that

$$f \epsilon^{\alpha\beta} = \bar{\epsilon}^\alpha \epsilon^\beta, \quad V^I \epsilon^{\alpha\beta} = \bar{\epsilon}^\alpha \Gamma^I \epsilon^\beta, \quad \Phi^{\alpha\beta IJ} = \bar{\epsilon}^\alpha \Gamma^{IJ} \epsilon^\beta. \quad (66)$$

We can find various algebraic relations between these bilinears by using the Fierz rearrangement identity for the product of four spinors  $\bar{\epsilon}_1, \epsilon_2, \bar{\epsilon}_3$  and  $\epsilon_4$ , given by

$$\bar{\epsilon}_1 \epsilon_2 \bar{\epsilon}_3 \epsilon_4 = \frac{1}{4} \left( \bar{\epsilon}_1 \epsilon_4 \bar{\epsilon}_3 \epsilon_2 + \bar{\epsilon}_1 \Gamma_I \epsilon_4 \bar{\epsilon}_3 \Gamma^I \epsilon_2 - \frac{1}{2} \bar{\epsilon}_1 \Gamma_{IJ} \epsilon_4 \bar{\epsilon}_3 \Gamma^{IJ} \epsilon_2 \right). \quad (67)$$

Of particular interest for our purposes is the identity with  $\bar{\epsilon}_1 = \bar{\epsilon}^\alpha$ ,  $\epsilon_2 = \epsilon^\delta$ ,  $\bar{\epsilon}_3 = \bar{\epsilon}^\gamma$  and  $\epsilon_4 = \epsilon^\beta$ . Then we find that

$$\epsilon^{\alpha\delta} \epsilon^{\gamma\beta} f^2 = \frac{1}{4} \epsilon^{\alpha\beta} \epsilon^{\gamma\delta} (f^2 + V_I V^I) - \frac{1}{8} \Phi^{\alpha\beta}_{IJ} \Phi^{\gamma\delta IJ}, \quad (68)$$

and contracting both sides of this equation with  $\epsilon_{\alpha\beta} \epsilon_{\gamma\delta}$  leads to

$$V_I V^I = f^2. \quad (69)$$

This implies that the vector  $V^I$  is either timelike (if  $f \neq 0$ ) or null (if  $f = 0$ ).

As for IHs in four dimensions, we are interested in the null case. Using the Killing spinor equation (65) and employing the Fierz identity, it can be shown that the covariant derivative of  $V^I$  is given by [53]

$$\nabla_I V_J = \frac{1}{2\sqrt{3}} \epsilon_{IJKLM} F^{KL} V^M. \quad (70)$$

Using this, we conclude that an IH of five-dimensional EMCS theory, equipped with a null normal  $\ell$ , will be supersymmetric if

$$\begin{aligned} \nabla_a \ell_b &= e_a^I \nabla_I (e_b^J \ell_J) \\ &= \frac{1}{2\sqrt{3}} e_a^I e_b^J \epsilon_{IJKLM} F^{KL} \ell^M; \end{aligned} \quad (71)$$

it is not difficult to see that the pull-back of this expression to  $\Delta$  vanishes because of the IH condition (16) on  $\mathbf{F}$  and the pullback expression (23) for  $e$ . Thus we again find that

$$\nabla_{\underline{a}} \ell_b \approx 0, \quad (72)$$

and with (14) it follows that  $\omega = 0$ . The existence of a Killing spinor on an IH in five-dimensional EMCS theory implies that the connection one-form is zero, as was found for the four-dimensional EM theory.

### 5.3 Interpretation

We come to the following conclusion. The necessary and sufficient condition for supersymmetry is that there exists a Killing spinor  $\psi$  in four dimensions that satisfies the conditions (51) and (52), or a Killing spinor  $\epsilon$  in five dimensions that satisfies (65). For an IH these conditions imply that the induced (normal) connection has to vanish. This implies that

$$\kappa_{(\ell)} = \omega_a \ell^a = 0 \quad \text{and} \quad \tilde{\omega}_a = \omega_a + \kappa_{(\ell)} n_a = 0. \quad (73)$$

The fact that  $\omega = 0$  means that the IH is nonrotating provided that there exists a global rotational Killing field  $\varphi$  on  $\mathcal{M}$  whose restriction to  $\Delta$  is  $\phi$ . Therefore we conclude that *an IH is supersymmetric only if it is extremal and nonrotating*. These conditions further imply that the topology constraint (48) for SIHs is given by

$$\oint_{\mathbb{S}^{D-2}} \tilde{\epsilon} \mathcal{R} = 2 \oint_{\mathbb{S}^{D-2}} \tilde{\epsilon} (T_{ab} \ell^a n^b + \Lambda). \quad (74)$$

We see that there are two possibilities for the topology of a SIH (when  $\Lambda \geq 0$ ). If  $\Lambda > 0$  and  $T_{ab} \ell^a n^b \geq 0$  or if  $\Lambda \geq 0$  and  $T_{ab} \ell^a n^b > 0$ , then  $\oint_{\mathbb{S}^{D-2}} \tilde{\epsilon} \mathcal{R} > 0$ . In this case the SIH is of positive Yamabe type. On the other hand, if  $\Lambda = 0$  and  $T_{ab} \ell^a n^b = 0$  then  $\oint_{\mathbb{S}^{D-2}} \tilde{\epsilon} \mathcal{R} = 0$ . In this case the SIH can have torus topology.

In four dimensions, using the IH conditions and the Maxwell field equations and employing the methodology of Lewandowski and Pawłowski [16], it can also be shown that the quantity  $T_{ab} \ell^a n^b$  has to be constant over the horizon cross sections and therefore implies that  $\mathcal{R}$  is constant. Unfortunately, it does not appear that this result holds in five dimensions because the quantity  $T_{ab} \ell^a n^b$  may vary on the horizon in general. However, if one could show that  $T_{ab} \ell^a n^b$  cannot vanish on the horizon, then this would be significant as it would rule out the torus topology.

The quasilocal picture that we have presented is in excellent agreement with the results that are known for stationary spacetimes [29, 51]. In that context a supersymmetric black hole also contains an extremal and nonrotating horizon. Extremality is a consequence of the BPS bounds being saturated, which then implies that there exists a Killing spinor. Nonrotation is then a consequence of the fact that a vector cannot be constructed in the neighbourhood of a supersymmetric Killing

horizon that is spacelike, as can be seen from the algebraic conditions (56) and (69). Therefore the spacetime of such a black hole cannot contain an ergoregion, which means that the horizon must be nonrotating.

In five dimensions, there are two supersymmetric solutions with the property that the bulk electromagnetic field contains angular momentum while the horizon is nonrotating. These are the BMPV black hole [32] and the EEMR black ring [52]. The BMPV black hole has one independent rotation parameter. This corresponds to a SIH with one angular momentum given by

$$\mathcal{J} = \frac{1}{8\pi G_5} \oint_{S^3} (\phi \lrcorner \mathbf{A}) \Phi . \quad (75)$$

The spacetime of the BMPV black hole is described by a nonrotating spherical horizon with angular momentum stored in the electromagnetic fields. The angular momentum of the horizon is nonzero [29]. Therefore some fraction of  $\mathcal{J}_{\text{EM}}$  is on  $\Delta$ . In addition, the distribution of angular momentum of the spacetime is such that there is a negative fraction behind the horizon as well. The net result is that the horizon geometry is that of a squashed three-sphere. These interesting properties do *not* contradict the idea that an IH is the inner boundary of the manifold. On the contrary, IHs with arbitrary distortions and rotations in their neighbourhoods have been recently studied using multipole moments [33]. When the angular momentum of the BMPV black hole is zero the solution reduces to the extremal RN solution in isotropic coordinates.

The EEMR black ring solution is the generalization of the BMPV black hole solution to the case where two independent rotation parameters are present. This corresponds to a SIH with two angular momenta given by

$$\mathcal{J}_j = \frac{1}{8\pi G_5} \oint_{S^1 \times S^2} (\phi_j \lrcorner \mathbf{A}) \Phi \quad (j \in \{1, 2\}) . \quad (76)$$

In addition, a black ring can also have dipole charges which are naturally defined on the horizon [55–57]. For an IH with ring topology a definition for the dipole charge  $\mathcal{P}$  can be realized by integrating the electromagnetic field strength plus the CS contribution over  $S^2$ :

$$\mathcal{P} = \frac{1}{2\pi} \int_{S^2} \star \Phi . \quad (77)$$

Charges of this type appear in the first law for a black ring [55, 56]. However, this is not the case with the first law (39) that we derived. This is probably due to the fact that the dipole charges  $\mathcal{P}$  are associated with the presence of *magnetic charge*, which we have not incorporated into our current framework. If we include magnetic charge, then we should find an additional term in the first law of the form  $\Upsilon \delta \mathcal{P}$  with  $\Upsilon$  the dipole potential at the horizon.

In this paper we focused on null Killing spinors that are defined at the horizon itself. However, the results obtained here will not be affected for black holes if we

assume that the spinors are defined globally. We note that there are many other solutions with such spinors that are defined for the entire spacetime, which do not describe black holes. These are known as pp-waves (plane-fronted gravitational waves with parallel rays) – spacetimes with vanishing expansion, shear and twist, and are a subset of a wider class of solutions that share this property, known as Kundt spacetimes. For details, see e.g. [58]. Given that a spinor is defined globally means that  $\ell$ , which is hypersurface orthogonal everywhere, is also defined globally. Moreover, these spacetimes can always be foliated by non-expanding horizons [17]. In the context of IHs, we can therefore regard supersymmetry as a condition that fixes the scaling of  $\ell$  because  $\omega = 0$  (independently of the IH condition  $[\mathcal{L}_\ell, \mathcal{D}] = 0$ ) and selects a preferred foliation of the manifold.

## 6 Prospects

In this paper we presented an extension of the IH framework to four-dimensional  $N = 2$  supergravity and five-dimensional  $N = 1$  supergravity. In particular, we derived the local version of the zeroth and first laws of black-hole mechanics for general WIHs on the phase space of the  $D$ -dimensional EMCS theory, and showed that SIHs have to be extremal and nonrotating. In the present work we focused mainly on mechanics of WIHs and the geometrical and topological constraints onto IHs that are imposed by supersymmetry and the field equations.

There are a number of classical applications of IHs to supergravity black holes that can be explored. Here we briefly discuss three problems that are worth investigating.

- *Asymptotically ADS spacetime.* Currently there is a lot of interest in the ADS/conformal field theory (ADS/CFT) correspondence [59, 60], and in particular in finding (supersymmetric) black hole solutions in ADS spacetime [30, 31, 61, 62]. It would therefore be worth while to study IHs of five-dimensional EMCS theory in asymptotically ADS spacetime. This would first require extending the covariant phase space that was constructed here to the phase space of Ashtekar *et al* [24], and imposing the appropriate fall-off conditions onto the fields at the boundary at infinity. In addition, the Killing spinor identity would need to be modified due to the presence of the cosmological constant, thus leading to a modification of the covariant derivative of the null normal. This would give the supersymmetry conditions for IHs in asymptotically ADS spacetime. It should be noted, however, that topological issues would need to be approached very carefully in this extension. This is because the topology of the black hole event horizon can be affected by the topology of the boundary at infinity [63]. This suggests that an investigation of the implications of topological censorship [64], if any, on IHs in asymptotically ADS spacetime is

necessary. Nevertheless, the resulting framework could provide us with new insights into the gravitational aspects of the ADS/CFT correspondence.

- *BPS bounds.* The general method for deriving the BPS bound for stationary spacetimes is to construct an expression for the energy of the spacetime using spinor identities and the Einstein field equations. This method was pioneered by Witten [65] and Nester [66] to prove the positive energy theorem. The method was applied in four dimensions [48] and in five dimensions [51] to calculate the BPS bounds for the corresponding spacetimes. How can one derive these bounds for IHs? The bounds are saturated when the spinors are supercovariantly constant, which is associated with extremality. This suggests that the extremality condition (47) can be used for IHs. This is straightforward to do for undistorted horizons. Let us consider the four-dimensional EM theory for illustration. Here the contraction  $T_{ab}\ell^a n^b$  is the square of the electric flux  $E_\perp$  crossing the surface [5]. For any IH this quantity is constant over  $S^2$  and can therefore be moved outside the integral. The result can be used to relate the charge  $Q$  on the horizon to its surface area  $A$  via  $Q = E_\perp A/(4\pi)$ . For the RN solution one finds that  $\eta = Q^2/R^2 - 1 \leq 0$  with  $R = \sqrt{A/(4\pi)}$  the areal radius [38]. When the surface gravity vanishes  $\eta = 0$  and  $Q = R = M$  with  $M$  the mass. This is the condition for supersymmetry in four dimensions. The situation is not as obvious for distorted horizons in five dimensions. This is because the contraction  $T_{ab}\ell^a n^b$ , which for EMCS theory is again the square of the electric flux, may not be constant over the horizon cross sections in general. However, for the BMPV black hole in particular we know that the cross sections are  $S^3$  which have constant curvature, and therefore  $E_\perp^2$  is constant on  $\Delta$ . From here, a charge-areal radius relation follows along the same lines as the derivation that was outlined above for the RN black hole in four dimensions.
- *Supersymmetry and black-hole uniqueness.* In four dimensions, it was shown [16] that the IH constraints for extremal IHs of EM theory are satisfied iff the intrinsic geometry of the horizon coincides with that of the extremal Kerr-Newman (KN) solution. An extension of that analysis to IHs of five-dimensional EMCS theory would be of interest, particularly because it would provide a method for deriving the geometries of the corresponding extremal IHs. This would complement a recently developed method [67, 68] for deriving the *near-horizon* geometries of extremal black holes. While speculating on the local uniqueness theorems in five dimensions we need to keep in mind that black holes in five dimensions are much less constrained than in four dimensions, mainly because in five dimension there are two possible topologies ( $S^3$  and  $S^1 \times S^2$ ), and also because there are two independent rotation parameters. As a consequence of this richer structure, it is possible that two distinct black

holes in five dimensions can have the same asymptotic charges [42]. This is a striking example of black-hole nonuniqueness in higher dimensions. Nevertheless, uniqueness has been established for supersymmetric black holes in five dimensions [69]. Therefore it is expected that the five-dimensional analogues of the local uniqueness theorem of [16] do exist, but for SIHs. We also note that, while supersymmetry constrains the geometry (i.e. connection one-form), the dominant energy condition and the Einstein field equations are still required to constrain the topology. Therefore we expect that there should be a unique horizon geometry for a given topology. For example, if the topology of a SIH is  $S^1 \times S^2$ , then the geometry should coincide uniquely with the induced metric and vector potential of the EEMR black ring solution. We also expect that, if the topology is  $S^3$ , then the geometry should coincide uniquely with that of the BMPV black hole *in general*, and the extremal RN black hole as a limiting case when the angular momentum of the Maxwell fields vanishes. It would be of considerable interest to try solving the IH constraints for a SIH when  $T_{ab}\ell^a n^b = 0$  (at the horizon); the resulting geometry would provide the first explicit solution of a supersymmetric black hole with toroidal topology.

The above are just three examples of how the IH framework can be applied to investigate the classical gravitational aspects of supergravity and superstring theory. The extension of IHs to supergravity originated with the objective of employing Hamiltonian methods to study aspects of the ADS/CFT correspondence. The extension of the work presented here to asymptotically ADS spacetimes and derivation of the corresponding Hamiltonian is currently in progress [70]. This will provide the point of departure for the study of quantum fields on null surfaces, with applications to black hole statistical mechanics in the ADS/CFT correspondence.

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